Journal of Approximation Theory **119**, 181–192 (2002)
doi:10.1006/jath.2002.3728

A Chebyshev Set and its Distance Function

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Communicated by Frank Deutsch

Received November 13, 2000; accepted August 19, 2002

We prove that in a Banach space X with rotund dual X^* a Chebyshev set C is convex iff the distance function d_C is regular on $X\backslash C$ iff d_C admits the strict and Gâteaux derivatives on $X\backslash C$ which are determined by the subdifferential $\partial||x-\bar{x}||$ for each $x \in X\backslash C$ and $\bar{x} \in P_C(x) := \{c \in C : ||x-c|| = d_C(x)\}$. If X is a reflexive Banach space with smooth and Kadec norm then C is convex iff it is weakly closed iff P_C is continuous. If the norms of X and X^* are Fréchet differentiable then C is convex iff d_C is Fréchet differentiable on $X\backslash C$. If also X has a uniformly Gâteaux differentiable norm then C is convex iff the Gâteaux (Fréchet) subdifferential $\partial^- d_C(x)$ $(\partial_F d_C(x))$ is nonempty on $X\backslash C$. © 2002 Elsevier Science (USA)

1. INTRODUCTION

Let X be a real normed linear space and X^* be its dual space. For a nonempty closed subset C in X, the *distance function* associated with C is defined as

$$d_C(x) = \inf\{||x-c|| : c \in C\} \qquad \forall x \in X$$

and a *minimizing sequence* for $x \in X$ is a sequence $\{x_n\} \subseteq C$ satisfying $||x_n - x|| \to d_C(x)$ as $n \to +\infty$. The *metric projection* is given by

$$P_C(x) = \{c \in C : ||x-c|| = d_C(x)\},$$

which consists of the closest points in C to $x \in X$. P_C is said to be continuous at $x \in X$ if $P_C(x)$ is a singleton and $y_n \to y \in P_C(x)$ whenever $y_n \in P_C(x_n)$ and $x_n \to x$ as $n \to +\infty$.

A nonempty subset C of X is said to be a *Chebyshev set* if each point in X has a unique closest point in C. (This concept was introduced by S. B. Stechkin in honour of the founder of best approximation theory, P. L. Chebyshev.) For example every nonempty closed convex set in a Hilbert

0021-9045/02 $35.00

space is a Chebyshev set. A Chebyshev set is necessarily closed. It is a well-known problem whether a Chebyshev set in a Hilbert space must be convex. In a finite-dimensional Hilbert space Bunt [10], Kritikos [24], and Jessen [21] proved that every Chebyshev set is convex. However, in an infinite-dimensional Hilbert space this problem is still open (see [1, 4, 15, 22]).

In a smooth space setting, many sufficient conditions for a Chebyshev set to be convex have also been obtained. Busemann [12] pointed out that every Chebyshev set in a smooth, strictly convex, and finite-dimensional space is convex. Klee [23] showed that in a finite-dimensional smooth normed linear space a Chebyshev set is convex. Efimov and Stechkin [17] proved that every weakly closed Chebyshev set in a smooth and uniformly rotund Banach space is convex (whose version in a Hilbert space is related to the work of Asplund [1] and Klee [23]) while Vlasov [25, 29] showed that every boundedly compact Chebyshev set in a smooth Banach space is convex. Vlasov also proved that in a Banach space which is uniformly smooth in each direction each approximately compact Chebyshev set is convex [26], and that in a strongly smooth space or in a Banach space X with rotund dual X^* every Chebyshev set with continuous metric projection is convex (see [27, 28, Theorem 3]). Further, Vlasov [29] gave the following sufficient condition for a Chebyshev set to be convex:

Theorem 1.1. *In a Banach space X with rotund dual X^*, a nonempty closed set C is convex if its distance function d_C satisfies*

$$D^+d_C(x) := \limsup_{\|y\|\to 0} \frac{d_C(x+y) - d_C(x)}{\|y\|} = 1$$

for all $x \in X\setminus C$. (*see also* [4, Proposition 2.1] *or* [5, Theorem 14]).

Balaganskii [2] indicated that in a strongly convex space X with Fréchet differentiable norm, if C is a Chebyshev set and A is the set of points $x \in X$ such that the relations $c_n \in C$ and $d(c_n, x) \to d_C(x)$ imply the existence of a convergent subsequence c_{n_k} and that the cardinality of the complement A^c is less than the cardinality of continuum, then C is convex. For more results in a smooth space see the excellent survey paper of Balaganskii and Vlasov [4] and the references therein.

Outside the smooth space setting, Brøndsted [7, 8] constructed, for each $n \geqslant 3$, a nonsmooth n-dimensional normed linear space in which every Chebyshev set is convex. He also proved that for each $n \leqslant 3$ every Chebyshev set in an n-dimensional normed linear space is convex if and only if each exposed point of the unit sphere is a smooth point. This was extended by Brown [9] in a four-dimensional normed linear space. In an almost smooth Banach space which generalizes smooth Banach spaces and nonsmooth

Banach spaces such as those Brøndsted constructed, Kanellopoulos has proved that every weakly compact Chebyshev set is convex [22].

It is worth noting that some of the above sufficient conditions for a Chebyshev set to be convex are also necessary. In the survey paper of Deutsch [15] we can see both a brief historical account of the convexity of a Chebyshev set and also interesting characterizations of a convex Chebyshev set in a Hilbert space. We prove in this paper that the sufficient condition in Theorem 1.1 is in fact necessary for a Chebyshev set to be convex. We also present many equivalent conditions in terms of various derivatives and subdifferentials often used in nonsmooth analysis. As a result, in a reflexive Banach space with smooth and Kadec norm, a Chebyshev set C is convex if and only if it is weakly closed.

For convenience we briefly review the following notions in nonsmooth analysis. Let $x \in X$ be fixed. For any $v \in X$, the *Clarke generalized directional derivative* of d_C at x in the direction v is

$$d_C{}^{\circ}(x; v) := \limsup_{\substack{y \to x \\ t \to 0^+}} \frac{d_C(y + tv) - d_C(y)}{t}$$

and the *Clarke subdifferential* of d_C at x is

$$\partial d_C(x) := \{\xi \in X^* : \langle \xi, v \rangle \leqslant d_C{}^{\circ}(x; v) \ \forall v \in X\}.$$

The *Michel–Penot generalized directional derivative* of d_C at x in the direction v is

$$d_C^{\diamondsuit}(x; v) := \sup_{u \in X} \limsup_{t \to 0^+} \frac{d_C(x + tu + tv) - d_C(x + tu)}{t}$$

and the *Michel–Penot generalized subdifferential* of d_C at x is

$$\partial^{\diamondsuit} d_C(x) := \{\xi \in X^* : \langle \xi, v \rangle \leqslant d_C^{\diamondsuit}(x; v) \ \forall v \in X\}.$$

Similarly, the *lower Dini derivative* of d_C at x in the direction v is

$$d_C^-(x; v) := \liminf_{t \downarrow 0} \frac{d_C(x + tv) - d_C(x)}{t}$$

and the *Gâteaux subdifferential* of d_C at x is the set

$$\partial^- d_C(x) := \{\xi \in X^* : \langle \xi, v \rangle \leqslant d_C^-(x; v) \quad \forall v \in X\}.$$

The *Fréchet subdifferential* of d_C at x is the set

$$\partial_F d_C(x) := \left\{ \xi \in X^* : \liminf_{y \to x} \frac{d_C(y) - d_C(x) - \langle \xi, y - x \rangle}{||y - x||} \geqslant 0 \right\}.$$

The *proximal subdifferential* of d_C at $x \in X$ is the set

$$\partial_P d_C(x) := \{ \xi \in X^* : \exists M > 0, \quad \delta > 0 \text{ s.t.}$$

$$d_C(y) - d_C(x) + M||y - x||^2 \geqslant \langle \xi, y - x \rangle \ \forall y \in x + \delta B \}.$$

The *upper Dini derivative* of d_C at x in the direction v is

$$d_C^+(x; v) := \limsup_{t \downarrow 0} \frac{d_C(x + tv) - d_C(x)}{t}$$

and the *usual directional derivative* of d_C at x in the direction v is

$$d'_C(x; v) := \lim_{t \downarrow 0} \frac{d_C(x + tv) - d_C(x)}{t}.$$

The *Gâteaux directional derivative* of d_C at x in the direction v is

$$Dd_C(x; v) := \lim_{t \to 0} \frac{d_C(x + tv) - d_C(x)}{t}.$$

The distance function d_C is said to be *strictly differentiable* at x if there exists $\xi \in X^*$ such that

$$\lim_{\substack{y \to x \\ t \to 0^+}} \frac{d_C(y + tv) - d_C(y)}{t} = \langle \xi, v \rangle$$

for each $v \in X$ and the convergence is uniform for v in compact sets. We say that d_C is *regular* at x if $d'_C(x; v)$ exists and equals $d_C{}^\circ(x; v)$ for every $v \in X$. d_C is said to be *Gâteaux differentiable* at $x \in X$ if there exists $\xi \in X^*$ such that $d'_C(x; v) = \langle \xi, v \rangle$ for each $v \in X$ and the convergence in the definition of $d'_C(x; v)$ is uniform with respect to v in finite sets. If the word "finite" in the preceding sentence is replaced with "bounded", then d_C is said to be *Fréchet differentiable* at x.

2. EQUIVALENT CONDITIONS WITH DERIVATIVES OF d_C

To establish our result we need the following two lemmas which characterize the strict and Gâteaux differentiabilities of the distance function on a smooth space.

LEMMA 2.1. *Suppose that the norm on a normed linear space X is smooth. Let C be a nonempty closed subset of X and let $x \in X \setminus C$ and $\bar{x} \in P_C(x)$. Then the following are equivalent*:

(i) *d_C is strictly differentiable at x.*

(ii) $\partial d_C(x) = \partial\|x - \bar{x}\|$.

Proof. Let d_C be strictly differentiable at x. Then $\partial d_C(x)$ is a singleton. Since X is smooth and $x \neq \bar{x}$, $\partial\|x - \bar{x}\|$ is also a singleton. To prove the implication (i) $\Rightarrow$ (ii) it suffices to show that $\partial d_C(x) \cap \partial\|x - \bar{x}\|$ is nonempty. Now by Borwein *et al.* [5, Theorem 5] there exists $\xi \in \partial^{\diamondsuit} d_C(x)$ such that $\xi \in \partial\|x - \bar{x}\|$, that is, $\partial^{\diamondsuit} d_C(x) \cap \partial\|x - \bar{x}\|$ is nonempty, which in turn implies that $\partial d_C(x) \cap \partial\|x - \bar{x}\|$ is nonempty since $\partial^{\diamondsuit} d_C(x) \subseteq \partial d_C(x)$.

Conversely, if $\partial d_C(x) = \partial\|x - \bar{x}\|$, then $\partial d_C(x)$ is a singleton since $\partial\|x - \bar{x}\|$ is. Therefore, by Clarke [13, Proposition 2.2.4], d_C is strictly differentiable at x. ∎

LEMMA 2.2. *Let C be a nonempty closed subset in a normed linear space X and $x \in X \setminus C$ with $\bar{x} \in P_C(x)$. Suppose that the norm of X is Gâteaux differentiable at $x - \bar{x}$. Then the following are equivalent*:

(i) *d_C is Gâteaux differentiable at x.*

(ii) $d_C^-(x; x - \bar{x}) = d_C(x)$.

(iii) $Dd_C(x; x - \bar{x}) = d_C(x)$.

Proof. (i) $\Rightarrow$ (ii) Suppose that the Gâteaux derivative $d'_C(x)$ exists. Then, by Burke *et al.* [11, Proposition 13] or Balaganskii [3, Theorem B], we have

$$d_C^-(x; x - \bar{x}) = \langle d'_C(x), x - \bar{x}\rangle = -\langle d'_C(x), \bar{x} - x\rangle = -d'_C(x; \bar{x} - x) = d_C(x).$$

Next we prove the implication (ii) $\Rightarrow$ (iii). Let $d_C^-(x; x - \bar{x}) = d_C(x)$. Using [11, Proposition 13] again, we have $d'_C(x; \bar{x} - x) = -d_C(x)$, so it suffices to show $d'_C(x; x - \bar{x}) = d_C(x)$. Now for any $t > 0$ there holds

$$d_C(x + t(x - \bar{x})) - d_C(x) \leqslant t\|x - \bar{x}\| = td_C(x)$$

from which it follows that $d_C^+(x; x - \bar{x}) \leqslant d_C(x)$. This together with

$$d_C^-(x; x - \bar{x}) = d_C(x)$$

implies that $d'_C(x; x - \bar{x})$ exists and equals $d_C(x)$. Thus (iii) holds.

Finally the implication (iii) $\Rightarrow$ (i) follows from [5, Corollary 2]. ∎

We are now in position to characterize a convex Chebyshev set in terms of various derivatives in nonsmooth analysis.

THEOREM 2.3. *Let C be a Chebyshev set in a Banach space X with rotund dual X^*. Then the following are equivalent*:

(A1) *C is convex.*

(A2) *d_C is convex.*

(B1) *d_C is regular on $X \backslash C$.*

(B2) *d_C is strictly differentiable on $X \backslash C$.*

(B3) *$d_C{}^\circ(x; \bar{x} - x) = -d_C(x)$ holds for each $x \in X \backslash C$ and $\bar{x} \in P_C(x)$.*

(B4) *$\partial d_C(x) = \partial ||x - \bar{x}||$ holds for each $x \in X \backslash C$ and $\bar{x} \in P_C(x)$.*

(C1) *d_C is Gâteaux differentiable on $X \backslash C$.*

(C2) *$d_C^-(x; x - \bar{x}) = d_C(x)$ holds for each $x \in X \backslash C$ and $\bar{x} \in P_C(x)$.*

(C3) *$Dd_C(x; x - \bar{x}) = d_C(x)$ holds for each $x \in X \backslash C$ and $\bar{x} \in P_C(x)$.*

(D1) *$Dd_C(x; v) = 1$ holds for each $x \in X \backslash C$ and some $v \in X$ with $||v|| = 1$.*

(D2) *$D^+ d_C(x) = 1$ holds for each $x \in X \backslash C$.*

Proof. The equivalence (A1) $\Leftrightarrow$ (A2) is obvious. Since the norm of X^* is rotund, X is smooth [16, Theorem 2, p. 23]. Thus the equivalences (B1) $\Leftrightarrow$ (B2) $\Leftrightarrow$ (B3) $\Leftrightarrow$ (B4) follow from Lemma 2.1 and [11, Theorem 16]. In addition, the equivalences (C1) $\Leftrightarrow$ (C2) $\Leftrightarrow$ (C3) and the implication (D2) $\Rightarrow$ (A1) are consequences of Lemma 2.2 and Theorem 1.1, respectively. It suffices to show the implications (A2) $\Rightarrow$ (B1), (B2) $\Rightarrow$ (C1) and (C3) $\Rightarrow$ (D1) $\Rightarrow$ (D2). But we note that the implication (B2) $\Rightarrow$ (C1) is from the definitions of strict and Gâteaux derivatives and that the implication (C3) $\Rightarrow$ (D1) is immediate by taking $v = (x - \bar{x})/||x - \bar{x}||$. It remains to prove the implications (A2) $\Rightarrow$ (B1) and (D1) $\Rightarrow$ (D2).

Now if (A2) holds, then, by Clarke [13, Proposition 2.3.6], d_C is regular at each point in X. Thus (B1) follows.

The implication (D1) $\Rightarrow$ (D2) is in fact immediate since for each $v \in X$ with $||v|| = 1$ we have $Dd_C(x; v) \leqslant D^+ d_C(x) \leqslant 1$. ∎

A norm on X is said to be *Kadec* if each weakly convergent sequence x_n in X with the weak limit $x \in X$ converges in norm to x whenever $||x_n|| \to ||x||$. Note that for a Chebyshev set C in a normed linear space X the metric projection P_C is continuous on $X \backslash C$ if and only if it is continuous on X which implies that $d'_C(x; x - \bar{x}) = d_C(x)$ for each $x \in X \backslash C$ and $\bar{x} \in P_C(x)$ (see [20, Corollary, p. 238]). In a reflexive Banach space X with smooth and Kadec norm, it is easy to verify that if a Chebyshev set C of X is weakly closed, then P_C is continuous on $X \backslash C$. Hence it must be convex.

THEOREM 2.4. *Let C be a Chebyshev set in a reflexive Banach space X with smooth and Kadec norm. Then the statements in Theorem* 2.3 *are equivalent to each of the following*:

(E1) *C is weakly closed.*

(E2) *P_C is continuous at each $x \in X \setminus C$.*

Proof. Recall that a reflexive Banach space X is smooth if and only if X^* is rotund (see [16, Corollary 1, p. 24]). Thus in a reflexive Banach space X with smooth and Kadec norm the statements in Theorem 2.3 are equivalent. As we explained above, statement (A1) in Theorem 2.3 follows from statement (E2).

Since a convex closed subset in a locally convex space is weakly closed (see [14, Corollary 1.5, p. 126]) and a Chebyshev set is always closed, a convex Chebyshev set must be weakly closed, that is, statement (A1) in Theorem 2.3 implies statement (E1). Hence it remains to show the implication (E1) $\Rightarrow$ (E2).

We suppose that C is weakly closed. For any $x \in X \setminus C$ and $\bar{x} \in P_C(x)$, consider any sequence $\{x_n\}$ with $\bar{x}_n \in P_C(x_n)$ and $x_n \to x$ as $n \to +\infty$. Note that d_C is Lipschitz continuous and $\bar{x}_n \in C$. We have

$$||x - \bar{x}|| \leqslant ||\bar{x}_n - x|| \leqslant ||x_n - \bar{x}_n|| + ||x_n - x|| = d_C(x_n) + ||x_n - x|| \to d_C(x).$$

This means that $\lim_{n \to +\infty} ||\bar{x}_n - x|| = ||x - \bar{x}||$ and hence $\{\bar{x}_n\}$ is bounded. Thus $\{\bar{x}_n\}$ is contained in an intersection set A of the weakly closed set C and a boundedly convex closed set and the set A is weakly closed. Since X is reflexive, by Conway [14, Theorem 4.2, p. 132], the ball $B = \{x \in X : ||x|| \leqslant 1\}$ is weakly compact. Thus the set A is weakly compact. Hence there exists a weakly convergent subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ whose weak limit $\bar{x}_0$ lies in A. Such an $\bar{x}_0$ must be in C. Note that the norm on a normed space is lower semicontinuous for the weak topology [14, p. 128]. Then

$$||x - \bar{x}_0|| \leqslant \liminf_{k \to +\infty} ||x - \bar{x}_{n_k}|| = d_C(x) = ||x - \bar{x}||.$$

This implies $\bar{x}_0 = \bar{x}$ since $P_C(x)$ is a singleton. And hence $x - \bar{x}_{n_k}$ weakly converges to $x - \bar{x}$ and satisfies $\lim_{k \to +\infty} ||x - \bar{x}_{n_k}|| = ||x - \bar{x}||$. Since the norm on X is Kadec, the sequence $x - \bar{x}_{n_k}$ is normly convergent to $x - \bar{x}$. Therefore, $\bar{x}_{n_k}$ converges to $\bar{x}$ in norm. This property means that every subsequence of $\{\bar{x}_n\}$ normly converges to $\bar{x}$. So $\bar{x}_n$ converges to $\bar{x}$ in norm. This proves that P_C is continuous at x since $x_n \to x$ is arbitrary. ∎

When the norms on Banach spaces X and X^* are Fréchet differentiable, Theorem 2.3 can be enriched with the following statements:

THEOREM 2.5. *Let the norms on Banach spaces X and X^* be Fréchet differentiable and C be a Chebyshev set of X. Then the statements in Theorem* 2.3 *are equivalent to each of the following*:

(E1) *P_C is continuous at each $x \in X \backslash C$.*

(E2) *Every minimizing sequence in C for each $x \in X \backslash C$ converges.*

(E3) *d_C is Fréchet differentiable on $X \backslash C$.*

If the norm of X is also Kadec, then the above statements are equivalent to each of the following:

(E4) *For each $x \in X \backslash C$ there exists $\bar{x} \in C$ such that every minimizing sequence $\{x_i\}$ in C for x weakly converges to $\bar{x}$.*

(E5) *C is weakly closed.*

Proof. Since the norm of X^* is Fréchet differentiable, the space X is reflexive ([16, Corollary 1, p. 34]). Note that the norm of X is smooth since it is Fréchet differentiable. Thus, by Diestel [16, Corollary 1, p. 24], X^* is rotund. Therefore for the present case the statements in Theorem 2.3 are equivalent and we only need to show the implications (D1) in Theorem 2.3 $\Rightarrow$ (E1), (E1) $\Rightarrow$ (E2) $\Rightarrow$ (E3), and (E3) $\Rightarrow$ (C1) in Theorem 2.3. But they easily follow from [19, Corollary 3.4; 18, Corollary 3.5], and the definitions of the Fréchet and Gâteaux derivatives, respectively.

Next if the norm of X is also Kadec, then by Theorem 2.4 the equivalence (E1) $\Leftrightarrow$ (E5) holds. It remains to show the equivalence (E2) $\Leftrightarrow$ (E4). For each $x \in X \backslash C$ we suppose (E2) holds, that is, all minimizing sequences in C for x are convergent. Then such sequences must converge to the same point in C which is denoted by $\bar{x}$. Thus (E4) holds since a normly convergent sequence must be weakly convergent.

Conversely, suppose that for each $x \in X \backslash C$ there exists $\bar{x} \in C$ such that every minimizing sequence $\{x_i\}$ in C for x weakly converges to $\bar{x}$. To prove the implication (E4) $\Rightarrow$ (E2), it suffices to show $x_i - x$ converges to $\bar{x} - x$ in norm. Since the norm of X is Kadec and the sequence $\{x_i - x\}$ weakly converges to $\bar{x} - x$, we only need to verify $\lim_{i\to+\infty} ||x_i - x|| = ||\bar{x} - x||$. Note that the norm is lower semicontinuous for the weak topology [14, p. 128] and $\bar{x} \in C$. We have

$$d_C(x) \leqslant ||\bar{x} - x|| \leqslant \liminf_{i\to+\infty} ||x_i - x|| = \lim_{i\to+\infty} ||x_i - x|| = d_C(x).$$

This proves $\lim_{i\to+\infty} ||x_i - x|| = ||\bar{x} - x||$ and hence completes the proof. ∎

3. EQUIVALENT CONDITIONS WITH SUBDIFFERENTIALS OF d_C

We recall that a norm $||\cdot||$ on X is said to be *uniformly Gâteaux differentiable* if for each nonzero point $x \in X$ there exists $\xi \in X^*$ such that

$$\lim_{t \downarrow 0} \frac{||x + tv|| - ||x||}{t} = \langle \xi, v \rangle \qquad \forall v \in X,$$

and for each $v \in X$ the convergence is uniform for all x with $||x|| = 1$. For a nonempty closed subset C in a normed linear space X with uniformly Gâteaux differentiable norm, Borwein, Fitzpatrick and Giles proved that $-d_C$ is regular at each $x \in X \setminus C$ (see [5, Theorem 8]). From this it follows that the Gâteaux differentiability of d_C at $x \in X \setminus C$ is equivalent to the nonemptiness of $\partial^- d_C(x)$.

LEMMA 3.1. *Let X be a normed linear space with uniformly Gâteaux differentiable norm and C be a nonempty closed subset in X. Then for any $x \in X \setminus C$ the distance function d_C is Gâteaux differentiable at x if and only if $\partial^- d_C(x)$ is nonempty.*

Proof. The necessity is obvious. To show the sufficiency we suppose that $\partial^- d_C(x)$ is nonempty for $x \in X \setminus C$. By Borwein *et al.* [5, Theorem 8], $-d_C$ is regular at x, so $(-d_C)^-(x; v) = (-d_C)^\circ(x; v)$ $\forall v \in X$. This implies that $(-d_C)^-(x; \cdot)$ is sublinear. And hence by the Hahn–Banach Theorem $\partial^-(-d_C)(x)$ is nonempty. Moreover, by definition we have

$$(-d_C)^-(x; v) + d_C^-(x; v) \leqslant 0 \qquad \forall v \in X,$$

so

$$\partial^-(-d_C)(x) + \partial^- d_C(x) \subseteq \{0\},$$

which together with the nonemptiness of $\partial^-(-d_C)(x)$ and $\partial^- d_C(x)$ implies that d_C is Gâteaux differentiable at x. ∎

Using Theorem 2.3 and Lemma 3.1, we characterize a convex Chebyshev set in terms of the Gâteaux subdifferential as follows.

THEOREM 3.2. *Let the norm on a Banach space X be uniformly Gâteaux differentiable and the norm of X^* be rotund. Then a Chebyshev set C in X is convex if and only if $\partial^- d_C(x)$ is nonempty for each $x \in X \setminus C$.*

Remark 3.3. From the proof of Lemma 3.1, the condition that the norm of X be uniformly Gâteaux differentiable in Theorem 3.2 can be replaced with $-d_C$ being regular on $X\setminus C$.

Note that $\partial_F d_C(x) \subseteq \partial^- d_C(x)$ for each $x \in X$. Under the condition of X and X^* in Theorem 3.2, the nonemptiness of $\partial_F d_C(x)$ for each $x \in X\setminus C$ is sufficient for a Chebyshev set C to be convex. This is also necessary when X is the same as in Theorem 2.5 and the norm of X is in addition uniformly Gâteaux differentiable or Kadec.

THEOREM 3.4. *Suppose that the norms on X and X^* are Fréchet differentiable and the norm of X is also uniformly Gâteaux differentiable. Then a Chebyshev set C in X is convex if and only if $\partial_F d_C(x)$ is nonempty for each $x \in X\setminus C$.*

Proof. If C is convex, then, by Theorem 2.5, d_C is Fréchet differentiable at each $x \in X\setminus C$. Thus $\partial_F d_C(x)$ is nonempty for each $x \in X\setminus C$.

Conversely, if $\partial_F d_C(x)$ is nonempty for each $x \in X\setminus C$, then $\partial^- d_C(x)$ is nonempty for each $x \in X\setminus C$. By Lemma 3.1 and Theorem 2.3, C is convex. ∎

For a nonempty closed set C in a reflexive Banach space X with Fréchet differentiable and Kadec norm, Borwein and Giles proved that if $\partial_F d_C(x)$ is nonempty for $x \in X\setminus C$, then d_C is Fréchet differentiable at x (see [6, Lemma 6]). Consequently we obtain the following result.

THEOREM 3.5. *Suppose that the norms on X and X^* are Fréchet differentiable and the norm of X is also Kadec. Then a Chebyshev set C in X is convex if and only if $\partial_F d_C(x)$ is nonempty for each $x \in X\setminus C$.*

Proof. This follows from Theorem 2.5 and [6, Lemma 6] since X is reflexive as stated in the proof of Theorem 2.5. ∎

Remark 3.6. According to Theorems 2.3, 3.2, 3.4 and 3.5, in a Hilbert space X, a Chebyshev set C is convex if and only if $\partial^- d_C(x) = \{\frac{x-\bar{x}}{\|x-\bar{x}\|}\}$ for each $x \in X\setminus C$ and $\bar{x} \in P_C(x)$ if and only if $\partial_F d_C(x) = \{\frac{x-\bar{x}}{\|x-\bar{x}\|}\}$ for each $x \in X\setminus C$ and $\bar{x} \in P_C(x)$ if and only if the Gâteaux derivative of d_C exists and equals $\frac{x-\bar{x}}{\|x-\bar{x}\|}$ for each $x \in X\setminus C$ and $\bar{x} \in P_C(x)$. In a finite-dimensional Hilbert space this statement is equivalent to those in the following result recently obtained by Wu [30], but in an infinite-dimensional Hilbert space we do not know if the equivalence is still true.

THEOREM 3.7. (Wu [30, Theorem 3.2]). *Let C be a nonempty closed subset in a Hilbert space X. Then the following are equivalent*:

(i) *C is a Chebyshev set.*

(ii) *For each $x \in X \setminus C$ there exists a unique $\xi \in X$ with $||\xi|| = 1$ such that*

$$\partial_P d_C(x + (t-1)d_C(x)\xi) = \{\xi\} \quad \forall t \in (0,1).$$

(iii) *For each $x \in X \setminus C$ there exists a unique $\xi \in X$ with $||\xi|| = 1$ such that*

$$\partial_F d_C(x + (t-1)d_C(x)\xi) = \{\xi\} \quad \forall t \in (0,1).$$

(iv) *For each $x \in X \setminus C$ there exists a unique $\xi \in X$ with $||\xi|| = 1$ such that the Fréchet derivative of d_C exists and equals ξ at $x + (t-1)d_C(x)\xi$ for all $t \in (0,1)$.*

(v) *For each $x \in X \setminus C$ there exists a unique $\xi \in X$ with $||\xi|| = 1$ such that*

$$\partial^- d_C(x + (t-1)d_C(x)\xi) = \{\xi\} \quad \forall t \in (0,1).$$

(vi) *For each $x \in X \setminus C$ there exists a unique $\xi \in X$ with $||\xi|| = 1$ such that the Gâteaux derivative of d_C exists and equals ξ at $x + (t-1)d_C(x)\xi$ for all $t \in (0,1)$.*

(vii) *For each $x \in X \setminus C$ there exists a unique $\xi \in X$ with $||\xi|| = 1$ such that the strict derivative of d_C exists and equals ξ at $x + (t-1)d_C(x)\xi$ for all $t \in (0,1)$.*

ACKNOWLEDGMENTS

The author is very grateful to Professor Deutsch and the referee of this paper for their many valuable comments and important suggestions.

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